

# Discrete approximations of determinantal point processes on continuous spaces: tree representations and tail triviality

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March 29, 2016

## Abstract

We prove tail triviality of determinantal point processes  $\mu$  on continuous spaces. Tail triviality had been proved for such processes only on discrete spaces, and hence we have generalized the result to continuous spaces. To do this, we construct tree representations, that is, discrete approximations of determinantal point processes enjoying a determinantal structure. There are many interesting examples of determinantal point processes on continuous spaces such as zero points of the hyperbolic Gaussian analytic function with Bergman kernel, and the thermodynamic limit of eigenvalues of Gaussian random matrices for Sine<sub>2</sub>, Airy<sub>2</sub>, Bessel<sub>2</sub>, and Ginibre point processes. Tail triviality of  $\mu$  plays a significant role in proving the uniqueness of solutions of infinite-dimensional stochastic differential equations (ISDEs) associated with  $\mu$ . For particle systems in  $\mathbb{R}$  arising from random matrix theory, there are two completely different constructions of natural stochastic dynamics. One is given by stochastic analysis through ISDEs and Dirichlet form theory, and the other is an algebraic method based on space-time correlation functions. Tail triviality is used crucially to prove the equivalence of these two stochastic dynamics.

## 1 Introduction

Let  $S$  be a locally compact, complete, separable metric space with metric  $d(\cdot, \cdot)$ . We assume  $S$  is unbounded. We equip  $S$  with a Radon measure  $\mathbf{m}$  such that  $\mathbf{m}(\mathcal{O}) > 0$  for any non-empty open set  $\mathcal{O}$  in  $S$ . Let  $\mathbf{S}$  be the configuration space over  $S$  (see (2.1) for definition).  $\mathbf{S}$  is a Polish space equipped with the vague topology.

A determinantal point process  $\mu$  on  $S$  is a probability measure on  $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$  for which the  $m$ -point correlation function  $\rho^m$  with respect to  $\mathbf{m}$  is given by the determinant

$$\rho^m(\mathbf{x}) = \det[K(x_i, x_j)]_{i,j=1}^m. \quad (1.1)$$

Here  $K: S \times S \rightarrow \mathbb{C}$  is a measurable kernel and  $\mathbf{x} = (x_1, \dots, x_m)$ . We refer to Section 2 and [1, 2, 20] for the definition of correlation functions and related notions.  $\mu$  is said to be associated with  $(K, \mathbf{m})$  and also a  $(K, \mathbf{m})$ -determinantal point process.

We set  $Kf(x) = \int_S K(x, y)f(y)\mathbf{m}(dy)$ . We regard  $K$  as an operator on  $L^2(S, \mathbf{m})$  and denote it by the same symbol. Throughout this paper, we assume:

(A1)  $K$  is Hermitian symmetric, of locally trace class, and  $\text{Spec}(K) \subset [0, 1]$ .

Here we say  $K$  is of locally trace class if  $K_A f(x) = \int 1_A(x) K(x, y) 1_A(y) f(y) \mathbf{m}(dy)$  is a trace class operator on  $L^2(S, \mathbf{m})$  for any compact set  $A$ .

From (A1) we deduce that the associated determinantal point process  $\mu = \mu^{K, \mathbf{m}}$  exists and is unique [20, 22, 9].

In the last two decades, determinantal point processes have been extensively studied. They contain many interesting examples; e.g., spanning trees and Schur measures on discrete spaces, zero points of the hyperbolic Gaussian analytic function with Bergman kernel, and thermodynamic limits of eigenvalues of Gaussian random matrices such as Sine<sub>2</sub>, Airy<sub>2</sub>, Bessel<sub>2</sub>, and Ginibre point processes on continuous spaces [1, 4, 20].

Determinantal point processes on discrete spaces have a well-behaved algebraic structure; as a result, some important facts are only known for discrete determinantal point processes [9, 10, 22]. One such example is tail triviality, which says that each event of a tail  $\sigma$ -field  $\text{Tail}(S)$  takes value 0 or 1. We refer to (2.3) for the definition of  $\text{Tail}(S)$ .

The purpose of this paper is to prove that the tail  $\sigma$ -field  $\text{Tail}(S)$  of  $S$  is trivial with respect to  $\mu$ . If the space  $S$  is discrete, then tail triviality has been proved by Shirai-Takahashi [21] for  $\text{Spec}(K) \subset (0, 1)$ , and by Russel Lyons [9] for  $\text{Spec}(K) \subset [0, 1]$ . If the space  $S$  is continuous, the problem remained open [10].

To prove tail triviality we introduce a discrete approximation for determinantal point processes, called the tree representation. This representation has a determinantal structure, and so belongs to determinantal point processes on discrete spaces.

A  $\mathbf{m}$ -partition  $\Delta = \{\mathcal{A}_i\}_{i \in I}$  of  $S$  is a countable collection of disjoint relatively compact, measurable subsets of  $S$  such that  $\cup_i \mathcal{A}_i = S$  and that  $\mathbf{m}(\mathcal{A}_i) > 0$  for all  $i \in I$ . For two partitions  $\Delta = \{\mathcal{A}_i\}_{i \in I}$  and  $\Gamma = \{\mathcal{B}_j\}_{j \in J}$ , we write  $\Delta \prec \Gamma$  if for each  $j \in J$  there exists  $i \in I$  such that  $\mathcal{B}_j \subset \mathcal{A}_i$ . We assume:

(A2) There exists a sequence of  $\mathbf{m}$ -partitions  $\{\Delta(\ell)\}_{\ell \in \mathbb{N}}$  satisfying (1.2)–(1.4).

$$\Delta(\ell) \prec \Delta(\ell + 1) \quad \text{for all } \ell \in \mathbb{N}, \quad (1.2)$$

$$\sigma\left[\bigcup_{\ell \in \mathbb{N}} \mathcal{F}_\ell\right] = \mathcal{B}(S), \quad (1.3)$$

$$\#\{j; \mathcal{A}_{\ell+1, j} \subset \mathcal{A}_{\ell, i}\} = 2 \text{ for all } i \in I(\ell) \text{ and } \ell \in \mathbb{N}, \quad (1.4)$$

where we set  $\Delta(\ell) = \{\mathcal{A}_{\ell, i}\}_{i \in I(\ell)}$  and  $\mathcal{F}_\ell := \mathcal{F}_{\Delta(\ell)} = \sigma[\mathcal{A}_{\ell, i}; i \in I(\ell)]$ .

Condition (1.4) is just for simplicity. This condition implies that the sequence  $\{\Delta(\ell)\}_{\ell \in \mathbb{N}}$  has a binary tree-like structure. We remark that (A2) is a mild assumption and, indeed, satisfied if  $S$  is an open set in  $\mathbb{R}^d$  and  $\mathbf{m}$  has positive density with respect to the Lebesgue measure. We now state one of our main theorems:

**Theorem 1.1.** *Assume (A1) and (A2). Let  $\mu$  be the  $(K, \mathbf{m})$ -determinantal point process. Then  $\mu$  has a trivial tail. That is,  $\mu(A) \in \{0, 1\}$  for all  $A \in \text{Tail}(S)$ .*

Many interesting determinantal point processes arise from random matrices such as Sine<sub>2</sub>, Airy<sub>2</sub>, and Bessel<sub>2</sub> point processes in  $\mathbb{R}$  and the Ginibre point process in  $\mathbb{R}^2$ . Applying Theorem 1.1 to these examples we obtain that all have trivial tails. We shall present these examples in Section 6.

One of our motivation for the focus on tail triviality of  $\mu$  is its relationship to the uniqueness of stochastic dynamics related to random matrices. Indeed, it plays a

significant role in proving the pathwise uniqueness of solutions of infinite-dimensional stochastic differential equations (ISDEs) related to  $\mu$  as indicated above [18]. Furthermore, it is also important for the coincidences in the stochastic dynamics given by the two different approaches, specifically, ISDEs and space-time correlation functions [19]. From the first approach, we obtain qualitative information of the dynamics, such as the semimartingale property and non-collision property for each tagged particle. From the second, we obtain quantitative information such as the moment bound of the linear statistics of particle systems. We explain these examples in Section 6.

We now explain the idea of the proof. We have two candidates for the discrete approximations of  $\mu$ . One is the approximation of the kernel  $K$ . Let  $K_\ell(x, y)$  be the discrete kernel on  $I(\ell)$  such that

$$K_\ell(x, y) = \frac{1}{m(\mathcal{A}_\ell(x))m(\mathcal{A}_\ell(y))} \int_{\mathcal{A}_\ell(x) \times \mathcal{A}_\ell(y)} K(u, v) m(du) m(dv),$$

where  $\mathcal{A}_\ell(x)$  is such that  $x \in \mathcal{A}_\ell(x) \in \Delta(\ell)$ . Then  $K_\ell$  can be regarded as a discrete kernel on  $I(\ell)$ . If  $K_\ell$  satisfies (A1), then  $K_\ell$  generates determinantal point field  $\mu_{K_\ell}$  (although the property  $\text{Spec}(K_\ell) \subset [0, 1]$  is non-trivial). One can expect the convergence of the kernel  $K_\ell$  to  $K$ , and as a result, the weak convergence of  $\mu_{K_\ell}$  to  $\mu$ , at least for continuous  $K$ . Because  $\mu_{K_\ell}$  is a determinantal point process on the discrete space, its tail  $\sigma$ -field is trivial. Such weak convergence, however, does not suffice for the convergence of the values on the tail  $\sigma$ -field  $\text{Tail}(\mathbf{S})$ .

Taking the above into account, we consider the second approximation given by  $\mu(\cdot|\mathcal{G}_\ell)$  below. Let  $\mathcal{G}_\ell$  be the sub- $\sigma$ -field of  $\mathcal{B}(\mathbf{S})$  given by

$$\mathcal{G}_\ell = \sigma[\{\mathbf{s} \in \mathbf{S}; \mathbf{s}(\mathcal{A}_{\ell,i}) = n\}; i \in I(\ell), n \in \mathbb{N}]. \quad (1.5)$$

Combining (1.2) and (1.3) with (1.5), we obtain

$$\mathcal{G}_\ell \subset \mathcal{G}_{\ell+1}, \quad \sigma[\mathcal{G}_\ell; \ell \in \mathbb{N}] = \mathcal{B}(\mathbf{S}). \quad (1.6)$$

Let  $\mu(\cdot|\mathcal{G}_\ell)$  be the regular conditional probability of  $\mu$  with respect to  $\mathcal{G}_\ell$ . Using (1.6), we shall prove in Lemma 5.3 that for all  $U \in \mathcal{B}(\mathbf{S})$

$$\lim_{\ell \rightarrow \infty} \mu(U|\mathcal{G}_\ell)(\mathbf{s}) = 1_U(\mathbf{s}) \quad \text{for } \mu\text{-a.s. } \mathbf{s}. \quad (1.7)$$

We see that the convergence in (1.7) is stronger than the weak convergence. In particular, the convergence in (1.7) is valid for all  $U \in \text{Tail}(\mathbf{S})$  because  $\text{Tail}(\mathbf{S}) \subset \mathcal{B}(\mathbf{S})$ .

We can naturally regard  $\Delta(\ell) = \{\mathcal{A}_{\ell,i}\}_{i \in I(\ell)}$  as a discrete, countable set with the interpretation that each element  $\mathcal{A}_{\ell,i}$  is a point. Thus,  $\mu(\cdot|\mathcal{G}_\ell)$  can be regarded as a point process on the discrete set  $\Delta(\ell)$ .

If  $\mu(\cdot|\mathcal{G}_\ell)$  would be a determinantal point process for each  $\ell$ , then Theorem 1.1 would follow from (1.7) immediately because determinantal point processes on discrete spaces always have trivial tails, and as discussed above,  $\mu(\cdot|\mathcal{G}_\ell)$  is naturally regarded as a determinantal point process on the discrete space  $\Delta(\ell)$ . This is clearly not the case because determinantal point processes are supported on single configurations and

$$\mu(\{\mathbf{s}; \mathbf{s}(\mathcal{A}_{\ell,i}) \geq 2\}|\mathcal{G}_\ell) > 0. \quad (1.8)$$

Hence we introduce a sequence of *fiber bundle-like sets*  $\mathbb{I}(\ell)$  ( $\ell \in \mathbb{N}$ ) in Section 2 with base space  $\Delta(\ell)$  with fiber consisting of a set of binary trees. We further expand

$\mathbb{I}(\ell)$  to  $\Omega(\ell)$  in (2.27), which has a fiber whose element is a product of a tree  $i$  and a component  $\mathcal{B}_{\ell,i}$  of partitions. See notation after Theorem 2.1.

Let  $\mu|_{\mathcal{G}_\ell}$  denote the restriction of  $\mu$  on  $\mathcal{G}_\ell$ . By construction  $\mu|_{\mathcal{G}_\ell}(\mathbf{A}) = \mu(\mathbf{A}|\mathcal{G}_\ell)$  for all  $\mathbf{A} \in \mathcal{G}_\ell$ . In Theorem 2.1 and Theorem 2.2, we construct a lift  $\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}$  of  $\mu|_{\mathcal{G}_\ell}$  on the fiber bundle  $\Omega(\ell)$ , and prove tail triviality of the lift  $\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}$  in Theorem 2.4, which establishes tail triviality of  $\mu|_{\mathcal{G}_\ell}$  in Theorem 2.5. Combining Theorem 2.5 with the martingale convergence theorem in Lemma 5.3, we obtain Theorem 1.1.

The key point of the construction of the lift  $\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}$  is that we construct a consistent family of orthonormal bases  $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$  in (2.15) and (2.16), and introduce the kernel  $\mathbf{K}_{\mathbb{F}(\ell)}$  on  $\mathbb{I}(\ell)$  in (2.21) such that

$$\mathbf{K}_{\mathbb{F}(\ell)}(i, j) = \int_{S \times S} \mathbf{K}(x, y) \overline{f_{\ell,i}(x)} f_{\ell,j}(y) \mathbf{m}(dx) \mathbf{m}(dy). \quad (2.21)$$

We shall prove in Lemma 3.2 that  $\mathbf{K}_{\mathbb{F}(\ell)}$  is a determinantal kernel on  $\mathbb{I}(\ell)$ , and present  $\nu_{\mathbb{F}(\ell)}$  as the associated determinantal point process on  $\mathbb{I}(\ell)$ . To some extent,  $\nu_{\mathbb{F}(\ell)}$  is a Fourier transform of  $\mu|_{\mathcal{G}_\ell}$  through the orthonormal basis  $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$ . We shall prove in Theorem 2.1 that their correlation functions  $\rho_{\mathcal{G}_\ell}^m$  and  $\rho_{\mathbb{F}(\ell)}^m$  satisfy a kind of Persival's identity:

$$\int_{\mathbb{A}} \rho_{\mathcal{G}_\ell}^m(\mathbf{x}) \mathbf{m}^m(d\mathbf{x}) = \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \rho_{\mathbb{F}(\ell)}^m(\mathbf{i}), \quad (2.26)$$

which is a key to construct the lift  $\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}$ .

While preparing the manuscript, we have heard that Professor A. Bufetov has proved independently tail triviality of determinantal point processes on continuous spaces independently from us (a seminar talk at Kyushu University in October 2015). His method is different from our own and requires a restriction on an integrability condition of the determinantal kernel  $\mathbf{K}(x, y)$ .

The organization of the paper is as follows. In Section 2, we give introduce definitions and concepts and state the main theorems (Theorems 2.1–2.5). We give tree representations of  $\mu$ . In Section 3, we prove Theorem 2.1. In Section 4, we prove Theorem 2.2–Theorem 2.5. In Section 5, we prove Theorem 1.1. In Section 6, we present motivational examples such as Sine<sub>2</sub>, Airy<sub>2</sub>, and Bessel<sub>2</sub>, and Ginibre point processes, and the related ISDEs. We explain the algebraic construction of stochastic dynamics associated with Sine<sub>2</sub>, Airy<sub>2</sub>, and Bessel<sub>2</sub> point processes.

## 2 Set up and main results

In this section, we recall various essentials and present the main theorems (Theorem 2.1–Theorem 2.5) other than Theorem 1.1.

A configuration space  $\mathbf{S}$  over  $S$  is a set consisting of configurations on  $S$  such that

$$\mathbf{S} = \{\mathbf{s}; \mathbf{s} = \sum_i \delta_{s_i}, \{s_i\} \subset S, \mathbf{s}(K) < \infty \text{ for any compact } K\}. \quad (2.1)$$

A probability measure  $\mu$  on  $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$  is called a point process, also called random point field. A symmetric function  $\rho^m$  on  $S^m$  is called the  $m$ -point correlation function of a point process  $\mu$  with respect to a Radon measure  $\mathbf{m}$  if it satisfies

$$\int_{\mathbf{S}} \prod_{i=1}^j \frac{\mathbf{s}(A_i)!}{(\mathbf{s}(A_i) - k_i)!} \mu(d\mathbf{s}) = \int_{A_1^{k_1} \times \dots \times A_j^{k_j}} \rho^m(\mathbf{x}) \mathbf{m}^m(d\mathbf{x}). \quad (2.2)$$

Here  $A_1, \dots, A_j \in \mathcal{B}(S)$  are disjoint and  $k_1, \dots, k_j \in \mathbb{N}$  such that  $k_1 + \dots + k_j = m$ . If  $\mathbf{s}(A_i) - k_i \leq 0$ , we set  $\mathbf{s}(A_i)! / (\mathbf{s}(A_i) - k_i)! = 0$ .

We fix a point  $o \in S$  as the origin, and set  $S_r = \{x \in S; d(o, x) < r\}$ . Each  $S_r$  is assumed to be relatively compact, and thus  $\mathbf{s}(S_r) < \infty$  for all  $\mathbf{s} \in \mathbf{S}$  and  $r \in \mathbb{N}$ . In this sense, each element  $\mathbf{s}$  of  $\mathbf{S}$  is a locally finite configuration. We note that this notion depends on the choice of metric  $d$  on  $S$ .

For a Borel set  $A$  we set  $\pi_A: \mathbf{S} \rightarrow \mathbf{S}$  by  $\pi_A(\mathbf{s})(\cdot) = \mathbf{s}(\cdot \cap A)$ . We set  $\pi_{S_r^c}: \mathbf{S} \rightarrow \mathbf{S}$  such that  $\pi_{S_r^c}(\mathbf{s}) = \mathbf{s}(\cdot \cap S_r^c)$ . We denote by  $\text{Tail}(\mathbf{S})$  the tail  $\sigma$ -field such that

$$\text{Tail}(\mathbf{S}) = \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}]. \quad (2.3)$$

If we replace  $S_r$  by any increasing sequence  $\{O_r\}$  of relatively compact open sets such that  $\bigcup_{r=1}^{\infty} O_r = S$ , then  $\text{Tail}(\mathbf{S})$  defines the same  $\sigma$ -field. Thus  $\text{Tail}(\mathbf{S})$  is independent of the choice of  $\{O_r\}$ .

Let  $\Delta(\ell) = \{\mathcal{A}_{\ell,i}\}_{i \in I(\ell)}$  be as in (A2), where  $\ell \in \mathbb{N}$ . We set  $\Delta = \{\mathcal{A}_i\}_{i \in I}$  such that

$$\Delta = \Delta(1), \quad \mathcal{A}_i = \mathcal{A}_{1,i} \quad I = I(1).$$

In consequence of (1.4), we assume without loss of generality that each element  $i$  of the parameter set  $I(\ell)$  is of the form

$$I(\ell) = I \times \{0, 1\}^{\ell-1}. \quad (2.4)$$

That is, each  $i \in I(\ell)$  is of the form  $i = (j_1, \dots, j_\ell) \in I \times \{0, 1\}^{\ell-1}$ . We take a label  $i \in \bigcup_{\ell=1}^{\infty} I(\ell)$  in such a way that, for  $\ell < \ell'$ ,  $i \in I(\ell)$ , and  $i' \in I(\ell')$ ,

$$\mathcal{A}_{\ell,i} \supset \mathcal{A}_{\ell',i'} \Leftrightarrow i = (j_1, \dots, j_\ell) \text{ and } i' = (j_1, \dots, j_\ell, \dots, j_{\ell'}).$$

We denote by  $\widetilde{\mathbb{I}}$  the set of all such parameters:

$$\widetilde{\mathbb{I}} = \sum_{\ell=1}^{\infty} I(\ell) = \sum_{\ell=1}^{\infty} I \times \{0, 1\}^{\ell-1}. \quad (2.5)$$

We can regard  $\widetilde{\mathbb{I}}$  as a collection of binary trees and  $I$  is the set of their roots.

**Example 2.1** (Binary partitions of  $\mathbb{R}$ ). Typically we can take  $S = \mathbb{R}$ ,  $\mathbf{m}(dx) = dx$ , and  $I = \mathbb{Z}$ . For  $i = (j_1, \dots, j_\ell) \in I(\ell)$ , we set  $J_{1,i} = j_1$  and, for  $\ell \geq 2$ ,

$$J_{\ell,i} = j_1 + \sum_{n=1}^{\ell-1} \frac{j_n}{2^n}. \quad (2.6)$$

We take  $\mathcal{A}_{\ell,i} = [J_{\ell,i}, J_{\ell,i} + 2^{-\ell+1})$ .

For  $i = (j_1, \dots, j_\ell) \in \widetilde{\mathbb{I}}$ , we set  $\text{rank}(i) = \ell$ . For  $i$  with  $\text{rank}(i) = \ell$ , we set

$$\mathcal{B}_i = \begin{cases} \mathcal{A}_{1,i} & \ell = 1, \\ \mathcal{A}_{\ell-1,i^-} & \ell \geq 2, \end{cases} \quad (2.7)$$

where  $i^- = (j_1, \dots, j_{\ell-1})$  for  $i = (j_1, \dots, j_\ell) \in I(\ell)$ . Let  $\mathbb{I} \subset \widetilde{\mathbb{I}}$  such that

$$\mathbb{I} = I + \sum_{\ell=2}^{\infty} \{i \in I(\ell); j_\ell = 0\}, \quad (2.8)$$

where  $i = (j_1, \dots, j_\ell) \in I(\ell)$ .

Let  $\mathbb{F} = \{f_i\}_{i \in \mathbb{I}}$  be an orthonormal basis of  $L^2(S, \mathbf{m})$  satisfying

$$\sigma[f_i; i \in \mathbb{I}, \text{rank}(i) = \ell] = \mathcal{F}_\ell \quad \text{for each } \ell \in \mathbb{N}, \quad (2.9)$$

$$\text{supp}(f_i) = \mathcal{B}_i \quad \text{for each } i \in \mathbb{I}, \quad (2.10)$$

$$f_i(x) = 1_{\mathcal{A}_i}(x) / \sqrt{\mathbf{m}(\mathcal{A}_i)} \quad \text{for } \text{rank}(i) = 1. \quad (2.11)$$

For a given sequence of  $\mathbf{m}$ -partitions satisfying (A2), such an orthonormal basis exists. We present here an example.

**Example 2.2** (Haar functions). We make the same assumptions as in Example 2.1. Let  $i = (j_1, \dots, j_\ell) \in \mathbb{I}$ . We set for,  $\ell = 1$  and  $i = (j_1)$ ,

$$f_i(x) = 1_{[j_1, j_1+1)}(x)$$

and, for  $\ell \geq 2$  and  $i = (j_1, \dots, j_\ell) \in \mathbb{I}$ ,

$$f_i(x) = 2^{(\ell-1)/2} \{1_{[J_{\ell,i}, J_{\ell,i}+2^{-\ell+1})}(x) - 1_{[J_{\ell,i}+2^{-\ell+1}, J_{\ell,i}+2^{-\ell+2})}(x)\}.$$

We can easily see that  $\{f_i\}_{i \in \mathbb{I}}$  is an orthonormal basis of  $L^2(\mathbb{R}, dx)$ . We remark that  $j_\ell = 0$  because  $i = (j_1, \dots, j_\ell) \in \mathbb{I}$  as we set in (2.8).

We next introduce the  $\ell$ -shift of above objects such as  $\mathbb{I}$ ,  $\mathcal{B}_i$ , and  $\mathbb{F} = \{f_i\}_{i \in \mathbb{I}}$ . Let  $\tilde{\mathbb{I}}(1) = \mathbb{I}$  and, for  $\ell \geq 2$ , we set

$$\tilde{\mathbb{I}}(\ell) := \sum_{r=1}^{\infty} I(\ell) \times \{0, 1\}^{r-1}, \quad (2.12)$$

where  $I(\ell) = I \times \{0, 1\}^{\ell-1}$  is as in (2.4). For  $\ell, r \in \mathbb{N}$ , we set  $\theta_{\ell-1,r} : \tilde{\mathbb{I}} \rightarrow \tilde{\mathbb{I}}(\ell)$  such that  $\theta_{0,r} = \text{id}$  ( $\ell = 1$ ) and, for  $\ell \geq 2$ ,

$$\theta_{\ell-1,r}((j_1, \dots, j_{\ell+r-1})) = (\mathbf{j}_\ell, j_{\ell+1}, \dots, j_{\ell+r-1}) \in I(\ell) \times \{0, 1\}^{r-1}, \quad (2.13)$$

where  $\mathbf{j}_\ell = (j_1, \dots, j_\ell) \in I(\ell)$ . For  $\ell = 1$ , we set  $\mathbb{I}(1) = \mathbb{I}$ . For  $\ell \geq 2$ , we set

$$\mathbb{I}(\ell) = I(\ell) + \sum_{r=2}^{\infty} \theta_{\ell-1,r}(\mathbb{I}). \quad (2.14)$$

We set  $\text{rank}(i) = r$  for  $i \in I(\ell) \times \{0, 1\}^{r-1}$ . By construction  $\text{rank}(i) = r$  for  $i \in \theta_{\ell-1,r}(\tilde{\mathbb{I}})$ . Let  $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$  such that, for  $r = \text{rank}(i)$ ,

$$f_{\ell,i}(x) = 1_{\mathcal{A}_{\ell,i}}(x) / \sqrt{\mathbf{m}(\mathcal{A}_{\ell,i})} \quad \text{for } r = 1, \quad (2.15)$$

$$f_{\ell,i}(x) = f_{\theta_{\ell-1,r}^{-1}(i)}(x) \quad \text{for } r \geq 2, \quad (2.16)$$

where  $\Delta(\ell) = \{\mathcal{A}_{\ell,i}\}_{i \in I(\ell)}$  is given in (A2). Then  $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$  is an orthonormal basis of  $L^2(S, \mathbf{m})$ . This follows from assumptions (2.15) and (2.16) and the fact that  $\mathbb{F} = \{f_i\}_{i \in \mathbb{I}}$  is an orthonormal basis.

*Remark 2.1.* (1) We note that  $f_{\ell,i} \in \mathbb{F}(\ell)$  is a newly defined function if  $\text{rank}(i) = 1$ , whereas  $f_{\ell,i} \in \mathbb{F}$  if  $\text{rank}(i) \geq 2$ . In particular, we see that

$$\{f_{\ell,i}\}_{i \in \mathbb{I}(\ell), \text{rank}(i) \geq 2} \subset \{f_i\}_{i \in \mathbb{I}, \text{rank}(i) \geq 2}. \quad (2.17)$$

(2) Let  $j = (j_1, \dots, j_{\ell+r-1}) \in \mathbb{I}$  and  $i = (\mathbf{j}_\ell, j_{\ell+1}, \dots, j_{\ell+r-1}) \in \mathbb{I}(\ell)$ . Then

$$j = \theta_{\ell-1,r}^{-1}(i).$$

Furthermore,  $f_{\ell,i} \in \mathbb{F}(\ell)$  and  $f_j \in \mathbb{F}$  satisfy  $f_{\ell,i} = f_j$  for  $r = \text{rank}(i) \geq 2$ .

(3) By construction, we see that

$$\sigma[f_{\ell,i}; i \in \mathbb{I}(\ell), \text{rank}(i) = r] = \mathcal{F}_{\ell-1+r} \quad \text{for each } \ell, r \in \mathbb{N}, \quad (2.18)$$

$$\text{supp}(f_{\ell,i}) = \mathcal{B}_{\ell,i} \quad \text{for all } i \in \mathbb{I}(\ell), \quad (2.19)$$

where we set, for  $j = \theta_{\ell-1,r}^{-1}(i)$  such that  $\text{rank}(i) = r$ ,

$$\mathcal{B}_{\ell,i} = \mathcal{B}_j. \quad (2.20)$$

Using the orthonormal basis  $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$ , we set  $\mathbf{K}_{\mathbb{F}(\ell)}$  on  $\mathbb{I}(\ell)$  by

$$\mathbf{K}_{\mathbb{F}(\ell)}(i, j) = \int_{S \times S} \mathbf{K}(x, y) \overline{f_{\ell,i}(x)} f_{\ell,j}(y) \mathbf{m}(dx) \mathbf{m}(dy). \quad (2.21)$$

Let  $\lambda_{\mathbb{I}(\ell)}$  be the counting measure on  $\mathbb{I}(\ell)$ . We shall prove in Lemma 3.2 that  $(\mathbf{K}_{\mathbb{F}(\ell)}, \lambda_{\mathbb{I}(\ell)})$  satisfies (A1). Hence we obtain the associated determinantal point process  $\nu_{\mathbb{F}(\ell)}$  on  $\mathbb{I}(\ell)$  from general theory [20, 22].

For  $i \in \mathbb{I}(\ell)$ , let  $\mathbf{m}_{f_{\ell,i}}(dx)$  be the probability measure on  $S$  such that

$$\mathbf{m}_{f_{\ell,i}}(dx) = |f_{\ell,i}(x)|^2 \mathbf{m}(dx). \quad (2.22)$$

For  $\mathbf{i} = (i_n)_{n=1}^m \in \mathbb{I}(\ell)^m$  and  $\mathbf{x} = (x_n)_{n=1}^m$ , where  $m \in \mathbb{N} \cup \{\infty\}$ , we set

$$\mathbf{m}_{f_{\ell,\mathbf{i}}}(d\mathbf{x}) = \prod_{n=1}^m |f_{\ell,i_n}(x_n)|^2 \mathbf{m}(dx_n). \quad (2.23)$$

By (2.16)  $\mathbf{m}_{f_{\ell,\mathbf{i}}}$  is a probability measure on  $S^m$ . By (2.19), we have

$$\mathbf{m}_{f_{\ell,\mathbf{i}}}(\prod_{n=1}^m \mathcal{B}_{\ell,i_n}) = 1. \quad (2.24)$$

Let  $\mathcal{G}_\ell$  be the sub- $\sigma$ -field as in (1.5). Let  $\nu_{\mathbb{F}(\ell)}$  be the  $(\mathbf{K}_{\mathbb{F}(\ell)}, \lambda_{\mathbb{I}(\ell)})$ -determinantal point process as before. Let  $\rho_{\mathcal{G}_\ell}^m$  and  $\rho_{\mathbb{F}(\ell)}^m$  be the  $m$ -point correlation functions of  $\mu|_{\mathcal{G}_\ell}$  and  $\nu_{\mathbb{F}(\ell)}$  with respect to  $\mathbf{m}$  and  $\lambda_{\mathbb{I}(\ell)}$ , respectively. We now state one of our main theorems:

**Theorem 2.1.** *Let  $\mathbb{I}_\ell(\mathcal{A}) = \{i \in \mathbb{I}(\ell); \mathcal{B}_{\ell,i} \subset \mathcal{A}\}$ . For  $\mathbb{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_m$ , we set*

$$\mathbb{I}_\ell(\mathbb{A}) = \mathbb{I}_\ell(\mathcal{A}_1) \times \dots \times \mathbb{I}_\ell(\mathcal{A}_m). \quad (2.25)$$

*Assume that  $\mathcal{A}_n \in \Delta(\ell)$  for all  $n = 1, \dots, m$ . Then*

$$\int_{\mathbb{A}} \rho_{\mathcal{G}_\ell}^m(\mathbf{x}) \mathbf{m}^m(d\mathbf{x}) = \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \rho_{\mathbb{F}(\ell)}^m(\mathbf{i}). \quad (2.26)$$

Let  $\mathbb{I}(\ell)$  be the single configuration space over  $\mathbb{I}(\ell)$ . We write  $i \in \mathbf{i}$  if  $\mathbf{i}(\{i\}) = 1$ . Each  $\mathbf{i} = \sum_{i \in \mathbf{i}} \delta_i \in \mathbb{I}(\ell)$  can be regarded as a subset of  $\mathbb{I}(\ell)$  by the correspondence of  $\mathbf{i}$  to  $\{i\}_{i \in \mathbf{i}}$ . Let

$$\Omega(\ell) := \bigcup_{i \in \mathbb{I}(\ell)} \{i\} \times \mathcal{B}_{\ell,i}. \quad (2.27)$$

Let  $\underline{\Omega}(\ell)$  be the single configuration space over  $\Omega(\ell)$ . Then by definition each element  $\omega \in \underline{\Omega}(\ell)$  is of the form  $\omega = \sum_{i \in \mathbf{i}} \delta_{(i, s_i)}$  such that  $s_i \in \mathcal{B}_{\ell,i}$ . Hence

$$\underline{\Omega}(\ell) \subset \{\omega = \sum_{i \in \mathbf{i}} \delta_{(i, s_i)}; \mathbf{i} = \sum_{i \in \mathbf{i}} \delta_i \in \mathbb{I}(\ell), s_i \in \mathcal{B}_{\ell,i}\}. \quad (2.28)$$

Let  $\mathbf{m}_{f_{\ell,i}}$  be as in (2.22). We set

$$\mathbf{m}_{\mathbb{F}(\ell)} = \prod_{i \in \mathbb{I}(\ell)} \mathbf{m}_{f_{\ell,i}}, \quad \mathbf{m}_{f_{\ell,i}} = \prod_{i \in \mathbf{i}} \mathbf{m}_{f_{\ell,i}}. \quad (2.29)$$

*Remark 2.2.* Let  $\mathbf{i} = (i_1, \dots, i_m)$  and  $\mathbf{i} = \sum_{n=1}^m \delta_{i_n} \equiv \sum_{i \in \mathbf{i}} \delta_i$ . By definition  $\mathbf{m}_{f_{\ell,i}}$  in (2.29) is a product measure on the product space  $\prod_{i \in \mathbf{i}} \mathcal{B}_{\ell,i}$  with (unordered) parameter  $i \in \mathbf{i}$ , whereas  $\mathbf{m}_{f_{\ell,i}}$  in (2.23) is a product measure on the product space  $\mathcal{B}_{i_1} \times \dots \times \mathcal{B}_{i_m}$  with (ordered) parameter  $\mathbf{i} = (i_1, \dots, i_m)$ .

We set  $\iota_{\ell} : \underline{\Omega}(\ell) \rightarrow \mathbb{I}(\ell)$  such that  $\iota_{\ell}(\omega) = \mathbf{i}$ , and  $\kappa_{\ell,i} : \underline{\Omega}(\ell) \rightarrow \prod_{i \in \mathbf{i}} \mathcal{B}_{\ell,i}$  such that  $\kappa_{\ell,i}(\omega) = (s_i)_{i \in \mathbf{i}}$ , where  $\omega = \sum_{i \in \mathbf{i}} \delta_{(i, s_i)}$ ,  $\mathbf{i} = \sum_{i \in \mathbf{i}} \delta_i$ , and  $s_i \in \mathcal{B}_{\ell,i}$ .

Let  $\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}$  be the probability measure on  $\underline{\Omega}(\ell)$  given by

$$(\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}) \circ \iota_{\ell}^{-1}(d\mathbf{i}) = \nu_{\mathbb{F}(\ell)}(d\mathbf{i}), \quad (2.30)$$

$$\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}(\kappa_{\ell,i}(\omega) \in d\mathbf{s} | \iota_{\ell}(\omega) = \mathbf{i}) = \mathbf{m}_{f_{\ell,i}}(d\mathbf{s}), \quad \mathbf{s} = (s_i)_{i \in \mathbf{i}}. \quad (2.31)$$

We can naturally regard the probability measures in (2.31) as a point process on  $\prod_{i \in \mathbf{i}} \mathcal{B}_{\ell,i}$  supported on the set of configurations with exactly one particle configuration  $\mathbf{s} = \delta_{\mathbf{s}}$  on  $\prod_{i \in \mathbf{i}} \mathcal{B}_{\ell,i}$ , that is,  $\mathbf{s} = (s_i)_{i \in \mathbf{i}}$  is such that  $s_i \in \mathcal{B}_{\ell,i}$  for each  $i \in \mathbf{i}$ .

**Theorem 2.2.** *Let  $\mathbf{u}_{\ell} : \underline{\Omega}(\ell) \rightarrow S$  be such that  $\mathbf{u}_{\ell}(\omega) = \sum_{i \in \mathbf{i}} \delta_{s_i}$ , where  $\omega = \sum_{i \in \mathbf{i}} \delta_{(i, s_i)}$ . Then*

$$\mu|_{\mathcal{G}_{\ell}} = (\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}) \circ \mathbf{u}_{\ell}^{-1}|_{\mathcal{G}_{\ell}}. \quad (2.32)$$

*Remark 2.3.* Theorem 2.2 implies that  $\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}$  is a *lift* of  $\mu|_{\mathcal{G}_{\ell}}$  onto  $\underline{\Omega}(\ell)$ . We can naturally regard  $\widetilde{\mathbb{I}}(\ell)$  as binary trees. Hence we call  $\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}$  a tree representation of  $\mu$  of level  $\ell$ .

We present a decomposition of  $\mu|_{\mathcal{G}_{\ell}}$ , which follows from Theorem 2.2 immediately. Let  $\mathbf{m}_{f_{\ell,i}}^{\mathbf{u}} = \mathbf{m}_{f_{\ell,i}} \circ \mathbf{u}_{\ell,i}^{-1}$ , where  $\mathbf{u}_{\ell,i} : \prod_{i \in \mathbf{i}} \mathcal{B}_{\ell,i} \rightarrow S$  is the unlabel map such that  $\mathbf{u}_{\ell,i}((s_i)_{i \in \mathbf{i}}) = \sum_{i \in \mathbf{i}} \delta_{s_i}$ .

**Theorem 2.3.** *For each  $A \in \mathcal{G}_{\ell}$ ,*

$$\mu(A) = \int_{\mathbb{I}(\ell)} \nu_{\mathbb{F}(\ell)}(d\mathbf{i}) \mathbf{m}_{f_{\ell,i}}^{\mathbf{u}}(A). \quad (2.33)$$

Let  $\mathbb{I}(\ell)_p = \{i \in \mathbb{I}(\ell); r \leq p, |j_1| \leq p\}$ , where  $i = (\mathbf{j}_{\ell}, j_{\ell+1}, \dots, j_{\ell+r-1})$ ,  $r = \text{rank}(i)$ , and  $\mathbf{j}_{\ell} = (j_1, j_2, \dots, j_{\ell})$ . Let  $\pi_p^c(i) = i(\cdot \cap \mathbb{I}(\ell)_p^c)$ . Then we set  $\text{Tail}(\mathbb{I}(\ell)) = \cap_{p=1}^{\infty} \sigma[\pi_p^c]$ . From this we can define the tail  $\sigma$ -field  $\text{Tail}(\underline{\Omega}(\ell))$  of  $\underline{\Omega}(\ell)$  because  $\Omega(\ell)$  is a subset of  $\mathbb{I}(\ell) \times S$ .



**Theorem 2.4.**  $\nu_{\mathbb{F}(\ell)} \diamond \mathfrak{m}_{\mathbb{F}(\ell)}$  is trivial on  $\text{Tail}(\underline{\Omega}(\ell)) \cap \mathfrak{u}_\ell^{-1}(\mathcal{G}_\ell)$ . That is,

$$\nu_{\mathbb{F}(\ell)} \diamond \mathfrak{m}_{\mathbb{F}(\ell)}(\mathbf{A}) \in \{0, 1\} \quad \text{for all } \mathbf{A} \in \text{Tail}(\underline{\Omega}(\ell)) \cap \mathfrak{u}_\ell^{-1}(\mathcal{G}_\ell). \quad (2.34)$$

We remark that  $\mu|_{\mathcal{G}_\ell}$  is *not* a determinantal point process. Hence we exploit  $\nu_{\mathbb{F}(\ell)} \diamond \mathfrak{m}_{\mathbb{F}(\ell)}$  instead of  $\mu|_{\mathcal{G}_\ell}$ . As we have seen in Theorem 2.2,  $\nu_{\mathbb{F}(\ell)} \diamond \mathfrak{m}_{\mathbb{F}(\ell)}$  is a lift of  $\mu|_{\mathcal{G}_\ell}$  in the sense of (2.32), from which we can deduce nice properties of  $\mu|_{\mathcal{G}_\ell}$ . Indeed, an application of Theorem 2.2 combined with Theorem 2.4 is tail triviality of  $\mu|_{\mathcal{G}_\ell}$ :

**Theorem 2.5.**  $\mu|_{\mathcal{G}_\ell}$  is tail trivial. That is,

$$\mu|_{\mathcal{G}_\ell}(\mathbf{B}) \in \{0, 1\} \quad \text{for all } \mathbf{B} \in \text{Tail}(\mathbf{S}) \cap \mathcal{G}_\ell. \quad (2.35)$$

We shall apply Theorem 2.5 to prove Theorem 1.1 in Section 5.

### 3 Proof of Theorem 2.1

The purpose of this section is to prove Theorem 2.1. In Lemma 3.1, we present a kind of Persival's identity of kernels  $\mathbf{K}$  and  $\mathbf{K}_{\mathbb{F}(\ell)}$  using the orthonormal basis  $\mathbb{F}(\ell)$ , where  $\mathbf{K}_{\mathbb{F}(\ell)}$  is the kernel given by (2.21) and  $\mathbb{F}(\ell)$  is as in (2.15) and (2.16). In Lemma 3.2, we prove  $(\mathbf{K}_{\mathbb{F}(\ell)}, \lambda_{\mathbb{I}(\ell)})$  is a determinantal kernel and the associated determinantal point process  $\nu_{\mathbb{F}(\ell)}$  exists. We will lift the Persival's identity between  $\mathbf{K}$  and  $\mathbf{K}_{\mathbb{F}(\ell)}$  to that of correlation functions of  $\mu|_{\mathcal{G}_\ell}$  and  $\nu_{\mathbb{F}(\ell)}$  in Theorem 2.1.

By definition  $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$  satisfies

$$\int_S |f_{\ell,i}(x)|^2 \mathfrak{m}(dx) = 1 \quad \text{for all } i \in \mathbb{I}(\ell), \quad (3.1)$$

$$\int_S f_{\ell,i}(x) \overline{f_{\ell,j}(x)} \mathfrak{m}(dx) = 0 \quad \text{for all } i \neq j \in \mathbb{I}(\ell). \quad (3.2)$$

**Lemma 3.1.** (1) Let  $P(x) = \sum_i \xi(i) f_{\ell,i}(x)$  and  $Q(y) = \sum_j \eta(j) f_{\ell,j}(y)$ . Suppose that the supports of  $\xi$  and  $\eta$  are finite sets. Then

$$\int_{S \times S} \mathbf{K}(x, y) \overline{P(x)} Q(y) \mathfrak{m}(dx) \mathfrak{m}(dy) = \sum_{i,j} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) \overline{\xi(i)} \eta(j). \quad (3.3)$$

(2) We have an expansion of  $\mathbf{K}$  in  $L^2_{\text{loc}}(S \times S, \mathfrak{m} \times \mathfrak{m})$  such that

$$\mathbf{K}(x, y) = \sum_{i,j \in \mathbb{I}(\ell)} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) f_{\ell,i}(x) \overline{f_{\ell,j}(y)}. \quad (3.4)$$

*Proof.* From (2.21) we deduce that

$$\begin{aligned} & \int_{S \times S} \mathbf{K}(x, y) \overline{P(x)} Q(y) \mathfrak{m}(dx) \mathfrak{m}(dy) \\ &= \int_{S \times S} \mathbf{K}(x, y) \sum_i \overline{\xi(i) f_{\ell,i}(x)} \sum_j \eta(j) f_{\ell,j}(y) \mathfrak{m}(dx) \mathfrak{m}(dy) \\ &= \sum_{i,j} \int_{S \times S} \mathbf{K}(x, y) \overline{f_{\ell,i}(x)} f_{\ell,j}(y) \mathfrak{m}(dx) \mathfrak{m}(dy) \overline{\xi(i)} \eta(j) \\ &= \sum_{i,j} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) \overline{\xi(i)} \eta(j). \end{aligned} \quad (3.5)$$

This yields (3.3). We have thus proved (1). By a direct calculation, we have

$$\begin{aligned} \int_S P(x) \overline{f_{\ell,i}(x)} \mathbf{m}(dx) &= \int_S \sum_p \xi(p) f_{\ell,p}(x) \overline{f_{\ell,i}(x)} \mathbf{m}(dx) = \xi(i), \\ \int_S Q(y) \overline{f_{\ell,j}(y)} \mathbf{m}(dy) &= \int_S \sum_q \eta(q) f_{\ell,q}(y) \overline{f_{\ell,j}(y)} \mathbf{m}(dy) = \eta(j). \end{aligned} \quad (3.6)$$

Combining (3.5) and (3.6) yields

$$\begin{aligned} \int_{S \times S} \mathbf{K}(x, y) \overline{P(x)} Q(y) \mathbf{m}(dx) \mathbf{m}(dy) &= \\ \int_{S \times S} \sum_{i,j} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) f_{\ell,i}(x) \overline{f_{\ell,j}(y)} \overline{P(x)} Q(y) \mathbf{m}(dx) \mathbf{m}(dy). \end{aligned}$$

This implies (3.4).  $\square$

Let  $\lambda_{\mathbb{I}(\ell)}$  be the counting measure on  $\mathbb{I}(\ell)$  as before. We can regard  $\mathbf{K}_{\mathbb{F}(\ell)}$  as an operator on  $L^2(\mathbb{I}(\ell), \lambda_{\mathbb{I}(\ell)})$  such that  $\mathbf{K}_{\mathbb{F}(\ell)} \xi(i) = \sum_{j \in \mathbb{I}(\ell)} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) \xi(j)$ . We now prove that the  $(\mathbf{K}_{\mathbb{F}(\ell)}, \lambda_{\mathbb{I}(\ell)})$ -determinantal point  $\nu_{\mathbb{F}(\ell)}$  process exists.

**Lemma 3.2.** *Let  $\text{Spec}(\mathbf{K}_{\mathbb{F}(\ell)})$  be the spectrum of  $\mathbf{K}_{\mathbb{F}(\ell)}$ . Then*

$$\text{Spec}(\mathbf{K}_{\mathbb{F}(\ell)}) \subset [0, 1]. \quad (3.7)$$

*In particular, there exists a unique, determinantal point process  $\nu_{\mathbb{F}(\ell)}$  on  $\mathbb{I}(\ell)$  associated with  $(\mathbf{K}_{\mathbb{F}(\ell)}, \lambda_{\mathbb{I}(\ell)})$ .*

*Proof.* We first prove (3.7). Let  $\xi$  be as in Lemma 3.1 and set

$$F(x) = \sum_{i \in \mathbb{I}(\ell)} \xi(i) f_{\ell,i}(x). \quad (3.8)$$

Recall that  $\mathbb{F}(\ell) = \{f_{\ell,i}\}_{i \in \mathbb{I}(\ell)}$  is an orthonormal basis. From this, (3.3) and (3.8), we deduce that

$$\int_{S \times S} \mathbf{K}(x, y) F(x) \overline{F(y)} \mathbf{m}(dx) \mathbf{m}(dy) = \sum_{i,j \in \mathbb{I}(\ell)} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) \xi(i) \overline{\xi(j)}, \quad (3.9)$$

$$\int_S |F(x)|^2 \mathbf{m}(dx) = \sum_{i \in \mathbb{I}(\ell)} |\xi(i)|^2. \quad (3.10)$$

From (A1) we obtain

$$0 \leq \int_{S \times S} \mathbf{K}(x, y) F(x) \overline{F(y)} \mathbf{m}(dx) \mathbf{m}(dy) \leq \int_S |F(x)|^2 \mathbf{m}(dx). \quad (3.11)$$

Combining (3.9) and (3.10) with (3.11), we obtain

$$0 \leq \sum_{i,j \in \mathbb{I}(\ell)} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) \xi(i) \overline{\xi(j)} \leq \sum_{i \in \mathbb{I}(\ell)} |\xi(i)|^2. \quad (3.12)$$

Because such  $\xi$  are dense in  $L^2(\mathbb{I}(\ell), \lambda_{\mathbb{I}(\ell)})$ , we obtain (3.7) from (3.12).

Because  $\mathbf{K}_{\mathbb{F}(\ell)}$  is Hermitian symmetric, the second claim is clear from (3.7) and general theory [20, 21, 22].  $\square$

**Lemma 3.3.** *Let  $\mathcal{B}_{\ell,i} = \text{supp}(f_{\ell,i})$  be as in (2.19). Then, for  $i, j \in \mathbb{I}(\ell)$  and  $\mathcal{A} \in \mathcal{F}_\ell$ ,*

$$\int_{\mathcal{A}} f_{\ell,i}(x) \overline{f_{\ell,j}(x)} \mathbf{m}(dx) = \begin{cases} 1 & (i = j, \mathcal{B}_{\ell,i} \subset \mathcal{A}) \\ 0 & (\text{otherwise}) \end{cases}. \quad (3.13)$$

*Proof.* We recall that  $\mathcal{B}_{\ell,i}$  is the support of  $f_{\ell,i}$  by (2.19).

Suppose  $i = j$  and  $\mathcal{B}_{\ell,i} \subset \mathcal{A}$ . Then from (3.1) we obtain

$$\int_{\mathcal{A}} f_{\ell,i}(x) \overline{f_{\ell,j}(x)} \mathbf{m}(dx) = \int_S f_{\ell,i}(x) \overline{f_{\ell,i}(x)} \mathbf{m}(dx) = 1. \quad (3.14)$$

Suppose that  $i = j$  and that  $\mathcal{B}_{\ell,i} \not\subset \mathcal{A}$ . Then, using  $\mathcal{A} \in \mathcal{F}_\ell$ , (2.7), and (2.20), we deduce that  $\mathcal{B}_{\ell,i} \cap \mathcal{A} = \emptyset$ . Because  $\mathcal{B}_{\ell,i} = \text{supp}(f_{\ell,i})$ , we obtain

$$\int_{\mathcal{A}} f_{\ell,i}(x) \overline{f_{\ell,j}(x)} \mathbf{m}(dx) = 0. \quad (3.15)$$

Finally, suppose  $i \neq j$ . Because  $\mathcal{A} \in \mathcal{F}_\ell$ , we see that  $\mathcal{B}_{\ell,i} \subset \mathcal{A}$  or  $\mathcal{B}_{\ell,i} \cap \mathcal{A} = \emptyset$ . The same also holds for  $\mathcal{B}_{\ell,j}$ . In any case, we obtain (3.15) from (3.2).

From (3.14) and (3.15), we obtain (3.13).  $\square$

*Proof of Theorem 2.1.* Because  $\mathcal{A}_n \in \Delta(\ell)$  for all  $n = 1, \dots, m$ , we have

$$\int_{\mathcal{A}_1 \times \dots \times \mathcal{A}_m} \rho_{\mathcal{G}_\ell}^m(\mathbf{x}) \mathbf{m}^m(d\mathbf{x}) = \int_{\mathcal{A}_1 \times \dots \times \mathcal{A}_m} \rho^m(\mathbf{x}) \mathbf{m}^m(d\mathbf{x}). \quad (3.16)$$

From (1.1) and (3.4) we deduce that

$$\rho^m(\mathbf{x}) = \det \left[ \sum_{i,j \in \mathbb{I}(\ell)} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) f_{\ell,i}(x_p) \overline{f_{\ell,j}(x_q)} \right]_{p,q=1}^m. \quad (3.17)$$

From Lemma 3.3, a straightforward calculation, and (2.22)–(2.24), we obtain

$$\begin{aligned} & \int_{\mathcal{A}_1 \times \dots \times \mathcal{A}_m} \det \left[ \sum_{i,j \in \mathbb{I}(\ell)} \mathbf{K}_{\mathbb{F}(\ell)}(i, j) f_{\ell,i}(x_p) \overline{f_{\ell,j}(x_q)} \right]_{p,q=1}^m \mathbf{m}^m(d\mathbf{x}) \\ &= \int_{\mathcal{A}_1 \times \dots \times \mathcal{A}_m} \sum_{(i_1, \dots, i_m) \in \mathbb{I}(\ell)} \det \left[ \mathbf{K}_{\mathbb{F}(\ell)}(i_p, i_q) \right]_{p,q=1}^m \prod_{r=1}^m |f_{\ell,i_r}(x_r)|^2 \mathbf{m}^m(d\mathbf{x}) \\ &= \int_{\mathcal{A}_1 \times \dots \times \mathcal{A}_m} \sum_{(i_1, \dots, i_m) \in \mathbb{I}(\ell)} \rho_{\mathbb{F}(\ell)}^m(i_1, \dots, i_m) \mathbf{m}_{f_{\ell,(i_1, \dots, i_m)}}(d\mathbf{x}) \\ &= \sum_{(i_1, \dots, i_m) \in \mathbb{I}_\ell(\mathcal{A}_1) \times \dots \times \mathbb{I}_\ell(\mathcal{A}_m)} \rho_{\mathbb{F}(\ell)}^m(i_1, \dots, i_m). \end{aligned} \quad (3.18)$$

Recall that  $\mathbb{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_m$ ,  $\mathbf{i} = (i_1, \dots, i_m)$ , and  $\mathbb{I}_\ell(\mathbb{A}) = \mathbb{I}_\ell(\mathcal{A}_1) \times \dots \times \mathbb{I}_\ell(\mathcal{A}_m)$  by (2.25). Combining these with (3.16)–(3.18) yields (2.26).  $\square$

## 4 Proof of Theorem 2.2–Theorem 2.5

### 4.1 Proof of Theorem 2.2

Let  $\varrho^m$  be the  $m$ -point correlation function of  $(\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{f_\ell}) \circ \mathbf{u}_\ell^{-1}|_{\mathcal{G}_\ell}$ . Then it suffices for (2.32) to prove

$$\rho_{\mathcal{G}_\ell}^m(\mathbf{x}) = \varrho^m(\mathbf{x}). \quad (4.1)$$

From (1.5) and  $\mathcal{F}_\ell = \sigma[\mathcal{A}_{\ell,i}; i \in I(\ell)]$ , we see that  $\rho_{\mathcal{G}_\ell}^m$  and  $\varrho^m$  are  $\mathcal{F}_\ell^m$ -measurable. Let  $m = m_1 + \dots + m_k$ . Let  $\mathbb{A} = \mathcal{A}_1^{m_1} \times \dots \times \mathcal{A}_k^{m_k} \in \Delta(\ell)^m$  such that  $\mathcal{A}_p \cap \mathcal{A}_q = \emptyset$  if  $p \neq q$ . Let  $\mathbf{i} = (i_n)_{n=1}^m = (\mathbf{i}_1, \dots, \mathbf{i}_k) \in \mathbb{I}(\ell)^m$  such that  $\mathbf{i}_n \in \mathbb{I}(\ell)^{m_n}$ . From Theorem 2.1, we see that

$$\int_{\mathbb{A}} \rho_{\mathcal{G}_\ell}^m(\mathbf{x}) \mathbf{m}^m(d\mathbf{x}) = \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \rho_{\mathbb{F}(\ell)}^m(\mathbf{i}). \quad (4.2)$$

By the definition of correlation functions, (2.30), and (2.31), we see that

$$\begin{aligned} \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \rho_{\mathbb{F}(\ell)}^m(\mathbf{i}) &= \int_{\mathbb{I}(\ell)} \prod_{n=1}^k \frac{\mathbf{i}(\mathbb{I}_\ell(\mathcal{A}_n))!}{(\mathbf{i}(\mathbb{I}_\ell(\mathcal{A}_n)) - m_n)!} \nu_{\mathbb{F}(\ell)}(d\mathbf{i}) \\ &= \int_{\mathcal{S}} \prod_{n=1}^k \frac{\mathbf{s}(\mathcal{A}_n)!}{(\mathbf{s}(\mathcal{A}_n) - m_n)!} (\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{f_\ell}) \circ \mathbf{u}_\ell^{-1}|_{\mathcal{G}_\ell}(d\mathbf{s}) \\ &= \int_{\mathbb{A}} \varrho^m(\mathbf{x}) \mathbf{m}^m(d\mathbf{x}). \end{aligned} \quad (4.3)$$

Combining (4.2) and (4.3), we deduce that

$$\int_{\mathbb{A}} \rho_{\mathcal{G}_\ell}^m(\mathbf{x}) \mathbf{m}^m(d\mathbf{x}) = \sum_{\mathbf{i} \in \mathbb{I}_\ell(\mathbb{A})} \rho_{\mathbb{F}(\ell)}^m(\mathbf{i}) = \int_{\mathbb{A}} \varrho^m(\mathbf{x}) \mathbf{m}^m(d\mathbf{x}). \quad (4.4)$$

From (4.4), we obtain (4.1). This completes the proof of Theorem 2.2.  $\square$

### 4.2 Proof of Theorem 2.3

Theorem 2.3 follows from Theorem 2.2 immediately.  $\square$

### 4.3 Proof of Theorem 2.4

It is known that determinantal point processes on discrete spaces are tail trivial [9, 21]. Hence  $\nu_{\mathbb{F}(\ell)}$  is tail trivial by Lemma 3.2.

Let  $\mathbf{u}_\ell$  be as in Theorem 2.2. Let  $\mathbf{A} \in \mathbf{u}_\ell^{-1}(\mathcal{G}_\ell)$ . Then there exists a  $\mathbf{B} \in \mathcal{B}(\mathbb{I}(\ell))$  such that  $\mathbf{A} = \mathbf{u}_\ell^{-1}(\mathbf{B})$ . From this we deduce that, for each  $\mathbf{A} \in \text{Tail}(\underline{\Omega}(\ell)) \cap \mathbf{u}_\ell^{-1}(\mathcal{G}_\ell)$ , there exists a  $\mathbf{B} \in \text{Tail}(\mathbb{I}(\ell))$  such that  $\mathbf{A} = \mathbf{u}_\ell^{-1}(\mathbf{B})$ . Hence from (2.30) we deduce

$$\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}(\mathbf{A}) = \nu_{\mathbb{F}(\ell)}(\mathbf{B}). \quad (4.5)$$

From (4.5) and tail triviality of  $\nu_{\mathbb{F}(\ell)}$  we deduce that

$$\nu_{\mathbb{F}(\ell)} \diamond \mathbf{m}_{\mathbb{F}(\ell)}(\mathbf{A}) \in \{0, 1\} \quad (4.6)$$

for each  $A \in \text{Tail}(\underline{\Omega}(\ell)) \cap \mathfrak{u}_\ell^{-1}(\mathcal{G}_\ell)$ . We easily see that  $\mathfrak{u}_\ell^{-1}(\mathcal{G}_\ell) \subset \sigma[\iota_\ell]$ . Hence

$$\text{Tail}(\underline{\Omega}(\ell)) \cap \mathfrak{u}_\ell^{-1}(\mathcal{G}_\ell) \subset \text{Tail}(\underline{\Omega}(\ell)) \cap \sigma[\iota_\ell]. \quad (4.7)$$

Combining (4.6) and (4.7) completes the proof of Theorem 2.4.  $\square$

#### 4.4 Proof of Theorem 2.5

Let  $B \in \text{Tail}(\mathbf{S}) \cap \mathcal{G}_\ell$ . Then we deduce that

$$\mathfrak{u}_\ell^{-1}(B) \in \text{Tail}(\underline{\Omega}(\ell)) \cap \mathfrak{u}_\ell^{-1}(\mathcal{G}_\ell).$$

Hence from Theorem 2.2 and Theorem 2.4, we deduce that

$$\mu(B) = \mu|_{\mathcal{G}_\ell}(B) = \nu_{\mathbb{F}(\ell)} \diamond \mathfrak{m}_{\mathbb{F}(\ell)}(\mathfrak{u}_\ell^{-1}(B)) \in \{0, 1\}.$$

This completes the proof.  $\square$

### 5 Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1.

**Lemma 5.1.** *Let  $X$  be a  $\text{Tail}(\mathbf{S})$ -measurable and integrable random variable. Then  $E^\mu[X|\mathcal{G}_\ell]$  is  $\text{Tail}(\mathbf{S}) \cap \mathcal{G}_\ell$ -measurable.*

*Proof.* Recall that  $\Delta(\ell) = \{\mathcal{A}_{\ell,i}\}_{i \in I(\ell)}$ . Let  $\pi_{T_r}$  be the projection with  $T_r$  such that

$$T_r = \bigcup_{\substack{\mathcal{A}_{\ell,i} \cap S_r \neq \emptyset; \\ i \in I(\ell)}} \mathcal{A}_{\ell,i}. \quad (5.1)$$

Then  $X \in L^1(\mathbf{S}, \mu)$  is  $\sigma[\pi_{T_r^c}]$ -measurable because  $X \in L^1(\mathbf{S}, \mu)$  is  $\text{Tail}(\mathbf{S})$ -measurable and each  $\mathcal{A}_{\ell,i}$  is relatively compact. Hence for each  $r \in \mathbb{N}$

$$X(\mathbf{s}) = X \circ \pi_{T_r^c}(\mathbf{s}). \quad (5.2)$$

From this we deduce that

$$E^\mu[X|\mathcal{G}_\ell] = E^\mu[X \circ \pi_{T_r^c}|\mathcal{G}_\ell]. \quad (5.3)$$

By construction  $S_r \subset T_r$ . Then from this and (5.3) we see that  $E^\mu[X|\mathcal{G}_\ell]$  is  $\sigma[\pi_{S_r^c}]$ -measurable for each  $r \in \mathbb{N}$ . Hence  $E^\mu[X|\mathcal{G}_\ell]$  is  $\text{Tail}(\mathbf{S})$ -measurable because  $\bigcap_{r \in \mathbb{N}} \sigma[\pi_{S_r^c}] = \text{Tail}(\mathbf{S})$ . By construction  $E^\mu[X|\mathcal{G}_\ell]$  is  $\bigcap_{r \in \mathbb{N}} \sigma[\pi_{S_r^c}]$ -measurable. Combining these completes the proof of Lemma 5.1  $\square$

**Lemma 5.2.** *For all  $A \in \text{Tail}(\mathbf{S})$*

$$\mu(A) = \mu(A|\mathcal{G}_\ell)(\mathbf{s}) \quad \text{for } \mu\text{-a.s. } \mathbf{s}. \quad (5.4)$$

*Proof.* From the definition of the conditional probability, we see that

$$\mu(A) = \int_{\mathbf{S}} \mu(A|\mathcal{G}_\ell)(\mathbf{s}) \mu(d\mathbf{s}). \quad (5.5)$$

From Lemma 5.1, we deduce that  $\mu(A|\mathcal{G}_\ell)(\mathbf{s}) = E^\mu[1_A|\mathcal{G}_\ell](\mathbf{s})$  is  $\text{Tail}(\mathbf{S}) \cap \mathcal{G}_\ell$ -measurable. Hence from Theorem 2.5 we obtain that  $\mu(A|\mathcal{G}_\ell)(\mathbf{s})$  is constant  $\mu$ -a.s.  $\mathbf{s}$ . This combined with (5.5) yields (5.4).  $\square$

**Lemma 5.3.** *For each  $A \in \mathcal{B}(S)$*

$$\lim_{\ell \rightarrow \infty} \mu(A|\mathcal{G}_\ell)(s) = 1_A(s) \quad \text{for } \mu\text{-a.s. } s. \quad (5.6)$$

*Proof.* From (1.6), we apply the martingale convergence theorem to obtain the convergence such that, for all  $A \in \mathcal{B}(S)$ ,

$$\lim_{\ell \rightarrow \infty} \mu(A|\mathcal{G}_\ell)(s) = \lim_{\ell \rightarrow \infty} E^\mu[1_A|\mathcal{G}_\ell](s) = E^\mu[1_A|\mathcal{B}(S)](s) = 1_A(s) \quad (5.7)$$

for  $\mu$ -a.s.  $s$ . We have thus proved (5.6).  $\square$

*Proof of Theorem 1.1.* From Lemma 5.2 and Lemma 5.3 we deduce that

$$\mu(A) = \mu(A|\mathcal{G}_\ell)(s) = 1_A(s) \quad \mu\text{-a.s. } s. \quad (5.8)$$

Hence we obtain  $\mu(A) \in \{0, 1\}$ .  $\square$

## 6 Examples related to random matrices and applications

In this section, we give typical examples of determinantal point processes related to random matrix theory. All examples below are tail trivial because of Theorem 1.1.

All the kernels  $K(x, y)$  below are continuous. In Examples 6.1–6.3, we define the kernels only off diagonal. On diagonal, they are defined by continuity. We refer to [18] for the precise meaning of ISDEs and the uniqueness of solutions.

We say that an  $S$ -valued process  $X = \{X_t\}$ , where  $X_t = \sum_i \delta_{X_t^i}$ , is *determinantal*, if the multi-time correlation functions for any chosen series of times are represented by determinants [19]. In other words, a *determinantal process* is an  $S$ -valued process such that, for any natural number  $M \in \mathbb{N}$ ,  $\mathbf{f} = (f_1, f_2, \dots, f_M) \in C_0(S)^M$ , and sequence of times  $\mathbf{t} = (t_1, t_2, \dots, t_M)$  with  $0 < t_1 < \dots < t_M < \infty$ , the moment generating function of multi-time distribution

$$\Psi^{\mathbf{t}}[\mathbf{f}] \equiv E \left[ \exp \left\{ \sum_{m=1}^M \int_S f_m(x) X_{t_m}(dx) \right\} \right]$$

is given by a Fredholm determinant

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \text{Det}_{\substack{(s,t) \in \{t_1, t_2, \dots, t_M\}^2 \\ (x,y) \in S^2}} \left[ \delta_{st} \delta(x-y) + \mathbb{K}(s, x; t, y) \chi_t(y) \right],$$

where  $\chi_{t_m} = e^{f_m} - 1$ ,  $1 \leq m \leq M$ , and  $\mathbb{K}$  is a locally integrable function called a (space-time) *correlation kernel* [5, 6, 7, 8]. Such a kernel exists for  $S$ -valued dynamics in one dimension related to random matrix theory and, in particular, the first three examples (Sine, Airy, and Bessel point processes) below. These are scaling limits of distributions of eigen values of random matrices at bulk, soft edge, and hard edge, respectively [1, 2, 11].

**Example 6.1** (sine point process). Let  $S = \mathbb{R}$  and  $\mathbf{m}(dx) = dx$ . Let

$$K_{\sin}(x, y) = \frac{\sin(x-y)}{\pi(x-y)} \quad (x \neq y)$$

be the sine kernel. Let  $\mu_{\sin}$  be the associated determinantal point process [11, 2].  $\mu_{\sin}$  is called the  $\text{sine}_2$  point process. From Theorem 1.1, we deduce that  $\mu_{\sin}$  is tail trivial.

The associated (labeled) stochastic dynamics is given by ISDE [13, 14, 18]:

$$dX_t^i = dB_t^i + \lim_{r \rightarrow \infty} \sum_{\substack{j \neq i \\ |X_t^i - X_t^j| < r}} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{Z}). \quad (6.1)$$

The ISDE (6.1) is called Dyson model in infinite dimensions [23]. The relation between point process  $\mu_{\sin}$  and ISDE (6.1) (and other pairs of point processes and ISDEs below) can be explained through the notion of logarithmic derivative of point process (see [13]). In [12], it is proved that particles of the above dynamics do not collide with each other. Hence, there exist a bijection between labeled dynamics  $\mathbf{X} = (X^i)$  and unlabeled dynamics  $\mathbf{X}$  if the initial label is given.

It is proved in [18] that (6.1) has pathwise, unique strong solutions for  $\mu_{\sin}$ -a.s. starting points. Furthermore, it is proved in [16, 19] that the associated  $\mathbf{S}$ -valued dynamics  $\mathbf{X} = \{\mathbf{X}_t\}$ , where  $\mathbf{X}_t = \sum_{i \in \mathbb{Z}} \delta_{X_t^i}$ , coincides with the determinantal process with kernel

$$\mathbb{K}_{\sin}(s, x; t, y) = \begin{cases} \frac{1}{\pi} \int_0^1 du e^{u^2(t-s)/2} \cos\{u(y-x)\} & \text{if } s < t, \\ K_{\sin}(x, y) & \text{if } s = t, \\ -\frac{1}{\pi} \int_1^\infty du e^{u^2(t-s)/2} \cos\{u(y-x)\} & \text{if } s > t. \end{cases}$$

Here  $s, t \in [0, \infty)$  and  $x, y \in \mathbb{R}$ . Tail triviality of  $\mu_{\sin}$  proved in Theorem 1.1 plays an important role for the pathwise uniqueness of solutions of ISDE (6.1) and the coincidence of these dynamics.

**Example 6.2** (Airy point process). Let  $\mathbf{S} = \mathbb{R}$  and  $\mathbf{m}(dx) = dx$ . Let

$$\mathbb{K}_{\text{Ai}}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y} \quad (x \neq y)$$

be the Airy kernel. Here  $\text{Ai}$  is the Airy function, and  $\text{Ai}'$  is its derivative. The associated determinantal point process  $\mu_{\text{Ai}}$  is called the Airy point process. It is proved in [15, 17] that the associated ISDE is given by

$$dX_t^i = dB_t^i + \lim_{r \rightarrow \infty} \left\{ \left( \sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right) - \int_{|x| < r} \frac{\hat{\rho}(x)}{-x} dx \right\} dt \quad (i \in \mathbb{N}).$$

Here we set

$$\hat{\rho}(x) = \frac{1_{(-\infty, 0)}(x)}{\pi} \sqrt{-x}.$$

Furthermore, it is proved in [16, 19] that the associated  $\mathbf{S}$ -valued dynamics  $\mathbf{X} = \{\mathbf{X}_t\}$ , where  $\mathbf{X}_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i}$ , coincides with the determinantal process with kernel

$$\mathbb{K}_{\text{Ai}}(s, x; t, y) = \begin{cases} \int_0^\infty du e^{-u(t-s)/2} \text{Ai}(u+x) \text{Ai}(u+y) & \text{if } s < t, \\ K_{\text{Ai}}(x, y) & \text{if } s = t, \\ -\int_{-\infty}^0 du e^{-u(t-s)/2} \text{Ai}(u+x) \text{Ai}(u+y) & \text{if } s > t. \end{cases}$$

Here  $s, t \in [0, \infty)$  and  $x, y \in \mathbb{R}$ .

**Example 6.3** (Bessel point process). Let  $S = [0, \infty)$  and  $\mathbf{m}(dx) = dx$ . Let  $1 \leq \alpha < \infty$ . Let  $K_{\text{Be},\alpha}$  be the Bessel kernel such that

$$K_{\text{Be},\alpha}(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})\sqrt{y}J_\alpha(\sqrt{y})}{2(x-y)} \quad (x \neq y).$$

Let  $\mu_{\text{Be},\alpha}$  be the associated determinantal point process.  $\mu_{\text{Be},\alpha}$  is called the Bessel $_{2,\alpha}$  point process. It is proved in [3] that the associated ISDE is given by

$$dX_t^i = dB_t^i + \left\{ \frac{\alpha}{2X_t^i} + \sum_{j \neq i} \frac{1}{X_t^i - X_t^j} \right\} dt \quad (i \in \mathbb{N}).$$

It is also proved in [16, 19] that the associated  $S$ -valued dynamics  $X_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i}$  coincides with the determinantal process with kernel

$$K_{\text{Be},\alpha}(s, x; t, y) = \begin{cases} \int_0^1 du e^{-2u(s-t)} J_\alpha(2\sqrt{ux}) J_\alpha(2\sqrt{uy}) & \text{if } s < t, \\ K_{J_\alpha}(x, y) & \text{if } s = t, \\ - \int_1^\infty du e^{-2u(s-t)} J_\alpha(2\sqrt{ux}) J_\alpha(2\sqrt{uy}) & \text{if } s > t. \end{cases}$$

Here  $s, t, x, y \in [0, \infty)$ .

**Example 6.4** (Ginibre point process). Let  $S = \mathbb{R}^2$  and  $\mathbf{m}(dx) = (1/\pi)e^{-|x|^2} dx$ . Let  $K_{\text{Gin}}: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{C}$  be the exponential kernel such that

$$K_{\text{Gin}}(x, y) = e^{x\bar{y}}.$$

Here we identify  $\mathbb{R}^2$  as  $\mathbb{C}$  by the obvious correspondence  $\mathbb{R}^2 \ni x = (x_1, x_2) \mapsto x_1 + \sqrt{-1}x_2 \in \mathbb{C}$ , and  $\bar{y} = y_1 - \sqrt{-1}y_2$  is the complex conjugate in this identification. Let  $\mu_{\text{Gin}}$  be the associated determinantal point process called the Ginibre point process. It is known that  $\mu_{\text{Gin}}$  is translation and rotation invariant.

We introduce two ISDEs [13, 18]. Both ISDEs are associated with  $\mu_{\text{Gin}}$ .

$$dX_t^i = dB_t^i + \lim_{r \rightarrow \infty} \sum_{|X_t^i - X_t^j| < r, j \neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N}).$$

$$dX_t^i = dB_t^i - X_t^i + \lim_{r \rightarrow \infty} \sum_{|X_t^j| < r, j \neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N}).$$

These ISDEs have pathwise unique, strong solutions [18]. One of the surprising facts is that these solutions are equal if their ( $S$ -valued) initial distributions are  $\mu_{\text{Gin}}$  and labeled initial distributions are given by the same label [13, 18]. The tail triviality of  $\mu_{\text{Gin}}$  plays a significant role for the proof of this result.

Unlike one dimensional particle systems as Examples 6.1–6.3, no explicit expressions of stochastic dynamics in terms of space-time correlation functions are known.



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**Acknowledgement:**

H.O. is supported in part by a Grant-in-Aid for Scientific Research (KIBAN-A, No. 24244010) from the Japan Society for the Promotion of Science.